A formula with hypervolumes of six 4-simplices and two discrete curvatures

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Abstract

One of the generalizations of the pentagon equation to higher dimensions is the so-called "six-term equation". Geometrically, it corresponds to one of the "Alexander moves", that is elementary rebuildings of simplicial complexes, namely, replacing a "cluster" of three 4-simplices by another "cluster", also of three 4-simplices and with the same boundary. We present a formula containing the euclidean volumes of the simplices in the first cluster in its l.h.s., and those in the second cluster — in its r.h.s. The formula also involves "discrete curvatures" appearing when we slightly deform the euclidean space.

As an artist and scholar I prefer the specific detail to the generalization, images to ideas, obscure facts to clear symbols, and the discovered wild fruit to the synthetic jam.

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1 Introduction

This short note continues the paper [1] and the short note [2]. We present a formula that naturally corresponds to one of the "Alexander moves" [3], i.e., "elementary rebuildings" of simplicial complexes. Our formula belongs to a four-dimensional space and deals with three "initial" 4-simplices in its l.h.s. and three "final" ones in is r.h.s.

Recall that a similar formula for a three-dimensional space was obtained in [2], and this was done on the basis of "duality formulas" (which are valid, themselves, for any-dimensional space) from [1][1].

2 Derivation of the formula

Consider six points A, B, C, D, E and F in the four-dimensional euclidean space[2]. There exist fifteen distances between them, which we denote, like in papers [1, 2], as

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[1]The duality in [1] can be said to be dealing with "branched polymers" known in quantum gravity.

[2]Yet, as will be seen below, we will allow them sometimes to "go out in the fifth dimension".

l_{AB}, l_{AC} and so on. There exist six 4-simplices with vertices in our points, and we denote those simplices as $\overline{A}$, …, $\overline{F}$, where, say, $\overline{A}$ is the simplex $BCDEF$ (*not containing* the vertex A). The *four-dimensional hypervolume* we will denote as V, e.g., $V_{\overline{A}}$ is the hypervolume of the simplex $\overline{A}$. We will need also the areas of *two-dimensional faces* (S_{ABC} being the area of face ABC and so on) and the "defect angles", or "discrete curvatures" concentrated in those faces (the defect angle ω_{ABC} corresponds to the face ABC, etc.).

A defect angle means the following. With *arbitrary* distances l_{AB}, …, l_{EF}, the points $A, \dots, F$ may not necessarily be placed in the ("flat") four-dimensional euclidean space. Any *five* of those points, however, *can* be placed there, thus forming a 4-simplex with vertices in those points, an then one can calculate the "dihedral angles" between its three-dimensional hyperfaces. There are three such "dihedral angles" at the two-dimensional face ABC — they correspond to tetrahedra $\overline{D}$, $\overline{E}$ and $\overline{F}$. In the flat case, the sum of those angles is 2π, and in the general case, it equals, by definition, $2\pi - \omega_{ABC}$.

All our considerations will take place in a small neighborhood of the flat case $\omega_{ABC} = 0$. Arguments perfectly analogous to those in [2], but using [1, formulas (15, 16)] instead of [1, formulas (11, 12)], yield

$$\frac{1}{12}\left|\frac{S_{ABC}\, l_{AB}\, dl_{AB}}{V_{\overline{D}}\, V_{\overline{E}}\, V_{\overline{F}}}\right| = \left|\frac{d\omega_{ABC}}{V_{\overline{A}}\, V_{\overline{B}}}\right|, \tag{1}$$

if only l_{AB} of all distances can vary. Similarly, one can write

$$\frac{1}{12}\left|\frac{S_{DEF}\, l_{DE}\, dl_{DE}}{V_{\overline{A}}\, V_{\overline{B}}\, V_{\overline{C}}}\right| = \left|\frac{d\omega_{DEF}}{V_{\overline{D}}\, V_{\overline{E}}}\right|, \tag{2}$$

if only l_{AB} can vary.

If both l_{AB} and l_{DE} can change, but the zero curvature is fixed:

$$\omega_{ABC} \equiv 0, \tag{3}$$

which is obviously equivalent to

$$\omega_{DEF} \equiv 0, \tag{4}$$

then dl_{AB} and dl_{DE} are related by

$$\left|\frac{l_{AB}\, dl_{AB}}{V_{\overline{D}}\, V_{\overline{E}}}\right| = \left|\frac{l_{DE}\, dl_{DE}}{V_{\overline{A}}\, V_{\overline{B}}}\right| \tag{5}$$

(cf. [1, (16)]), and similarly one can write out the relation between the differentials of any pair of distances.

Consider ω_{ABC} as a function of fifteen distances. In a neighborhood of the flat configuration, we have

$$d\omega_{ABC} = c_{AB}\, dl_{AB} + \cdots + c_{EF}\, dl_{EF}, \tag{6}$$

where all the ratios of coefficients $c_{\dots}$ are fixed (at least, up to a sign) by the formula (5) and the like formulae for other pairs of distances, e.g.,

$$\left|\frac{c_{AB}}{c_{DE}}\right| = \left|\frac{l_{AB}\, V_{\overline{A}}\, V_{\overline{B}}}{l_{DE}\, V_{\overline{D}}\, V_{\overline{E}}}\right|, \tag{7}$$

etc. If now we write, analogously,

$$d\omega_{DEF} = c'_{AB}\, dl_{AB} + \cdots + c'_{EF}\, dl_{EF}, \tag{8}$$

then the coefficients $c'_{\dots}$ will, obviously, have the *same* ratios. Thus, the differentials of curvatures ω_{ABC} and ω_{DEF} as functions of *all* distances are proportional, namely, from (1, 2 and 5) we find:

$$\left|\frac{d\omega_{ABC}}{S_{ABC}\, V_{\overline{A}}\, V_{\overline{B}}\, V_{\overline{C}}}\right| = \left|\frac{d\omega_{DEF}}{S_{DEF}\, V_{\overline{D}}\, V_{\overline{E}}\, V_{\overline{F}}}\right|. \tag{9}$$

Let us, finally, write this in the form of the desired “six-term equation”:

$$\frac{S_{ABC}\, \delta(\omega_{ABC})}{V_{\overline{D}}\, V_{\overline{E}}\, V_{\overline{F}}} = \frac{S_{DEF}\, \delta(\omega_{DEF})}{V_{\overline{A}}\, V_{\overline{B}}\, V_{\overline{C}}}. \tag{10}$$

Here δ is the Dirac delta function, and instead of writing out the absolute value signs, we assume that the signs of (oriented) hypervolumes and areas are chosen “properly”. It is implied that both sides of (10) can be integrated in any of $dl_{AB}, \dots, dl_{EF}$.

3 Remarks

1. Our equation (10) corresponds to a “move of type $3 \to 3$”, i.e., three simplices are transformed into three new ones. Nontrivial seems the question of what to do with the other Alexander moves, that is $2 \leftrightarrow 4$ and $1 \leftrightarrow 5$. Similarly, in the paper [2] the moves $2 \leftrightarrow 3$ are analyzed, while $1 \leftrightarrow 4$ requires further investigation.

2. Our formulas are likely to be useful for quantum gravity. Namely, they may help to find the most symmetric integration measure for “functional integrals” in the discrete Regge-type models of space-time.

3. The triangulated manifold where our rebuildings take place is not bound to be flat — see the similar Remark 2 in the end of paper [2].

References

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